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Sums over Partitions into Distinct Parts

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Abstract— In this paper, we are going to present sums over partitions. We define a function $u_n(g;q)$, which can be considered as one generalization of q-Pochhammer symbol. Using this function $u_n(g;q)$, many results have been proved for sums of partitions into even distinct parts, odd distinct parts, 1 as the only part and $v=v$ as a part.

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1. INTRODUCTION

Definition 1.1. *Riemann zeta function is defined by the convergent series*

$$\zeta(\kappa)=\sum_{v=1}^{\infty} v^{-\kappa}=\frac{1}{1^{\kappa}}+\frac{1}{2^{\kappa}}+\frac{1}{3^{\kappa}}+\cdots+\frac{1}{v^{\kappa}}+\cdots \qquad Re\ \kappa>1. \tag{1.1}$$

Initially, Euler [3] gave the product formula of the zeta function for the real variable, but Riemann [6] viewed zeta function as a function of a complex variable. One of its applications is that it connects complex analysis to number theory. Recently, Singh [8] discussed the values of the Riemann zeta function for integers.

The number theoretic properties are inherent in $\zeta(\kappa)$. These are given by the following formulae

$$\zeta(\kappa)=\sum_{v=1}^{\infty}\frac{1}{v^{\kappa}}=\prod_{p_v}(1-p_v^{-\kappa})^{-1} \tag{1.2}$$

where the product is taken over all prime numbers $p_v=2,3,5,7,\cdots$.

Definition 1.2. *A positive integer v can be written as a sum of positive integers as,*

$$v=v_1+v_2+\cdots+v_k$$

with $v_1\geq v_2\geq\cdots\geq v_k$ (and $k\geq 1$), then the number of ways to write the sum of above form of a positive integer is called the partition function and denoted by $p(v)$. We call the integers v_j's as the parts of the partition. It's not necessary for the parts to be distinct, and two partitions are same if they differ only in the order of their parts. Consider $p(0)$=1.

For examples

$$\begin{aligned} 3 &= 3 = 2+1 = 1+1+1, \qquad p(3) = 3. \\ 4 &= 4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1, \qquad p(4) = 5. \end{aligned}$$

Euler [3] discovered the generating function for the partition function which is given as

$$\sum_{v=0}^{\infty} p(v)q^v = \prod_{v=1}^{\infty}(1-q^v)^{-1} = \frac{1}{(q;q)_\infty} \tag{1.3}$$

Here, in keeping with the spirit of Euler, we take on the following kind of sum over partitions

$$\sum_{\mu\in\mathcal{P}} g(\mu).$$

Sums of the above form carry out many fascinating transformations and often reveal some special patterns with sums over natural numbers. In 2016, Schneider [7] established partition-theoretic formulas using q series, infinite products, and sums over integer partitions. He made connections between sums over partitions and many number-theoretic concepts, including the Riemann zeta function and multiple zeta values. In Section 3, we define a function $u_n(g;q)$, which can be considered as one generalization of the q-Pochhammer symbol. By using this function $u_n(g;q)$, two theorems have been proved that result in a sum over partitions into even distinct parts. While in Section 4, three theorems have been proved by considering sum over partitions as odd distinct parts, 1 as the only part, and $v = v$ as a part.

2. NOTATIONS

For $|q| < 1$, the following q notations are used:

$$(\omega;q)_n = (1-\omega)(1-\omega q)(1-\omega q^2)\cdots(1-\omega q^n), \qquad (\omega;q)_0 = 1. \tag{2.1}$$

$$(\omega;q)_\infty = (1-\omega)(1-\omega q)(1-\omega q^2)\cdots . \tag{2.2}$$

$$(q;q)_n = (1-q)(1-q^2)(1-q^3)\cdots(1-q^n). \tag{2.3}$$

$$(q;q)_\infty = (1-q)(1-q^2)(1-q^3)\cdots \tag{2.4}$$

The infinite series of the form

$$1 + q + q^2 + q^3 + \cdots = (1-q)^{-1} \tag{2.5}$$

is convergent for $|q| < 1$.

We also used the following notations and terminology:

If $\mu_1, \mu_2, \cdots, \mu_r$ are the integer parts of a partition μ, then

- The number of parts of μ is r. It is called the **length of partition** and denoted by $l(\mu)$, so $l(\mu) = r$.
- Sum of all the parts is called **size** or **weight of the partition**, and denote by $|\mu|$, i.e., $|\mu| = \mu_1 + \mu_2 + \cdots + \mu_r$. Also, μ is a partition of v is denoted by $\mu \vdash v$, and $|\mu| = v$.
- The product of parts of μ is called the **integer** of μ, and denoted by v_μ, i.e., $v_\mu = \mu_1\mu_2\cdots\mu_r$. Conventionally, $l(\emptyset) = 0$, $|\emptyset| = 0$, and $v_\emptyset = 1$.

3. SUMS OVER PARTITIONS AS EVEN DISTINCT PARTS

Before stating the theorems for sums over partitions as even distinct parts, we have to define $u_n(g;q)$.

For a function $g:\mathbb{N}\to\mathbb{C}$, we define $u_n(g;q)$ as

$$u_n(g;q)=\prod_{v=1}^{n}(1-g(v)q^v),\quad n\geq 1,\qquad u_0(g;q)=1. \tag{3.1}$$

If the product $\prod_{v=1}^{\infty}(1-g(v)q^v)$ converges, then

$$u_\infty(g;q)=\lim_{n\to\infty}\ u_n(g;q).$$

Hence, u can be considered as a generalization of q-Pochhammer symbol. Setting $g=$ a constant w, we get

$$u_n(w;q)=(wq;q)_n\ ,\qquad u_\infty(w;q)=(wq;q)_\infty.$$

In particular, if $g(v)=1$, then (3.1) reduces to (2.3).

Thus,

$$u_{e,\infty}(g;q)=\prod_{v=1}^{\infty}\left(1-g(2v)q^{2v}\right),$$

and

$$u_{o,\infty}(g;q)=\prod_{v=1}^{\infty}(1-g(2v-1)q^{2v-1}).$$

Since $1/u_\infty(g;q)=\prod_{v=1}^{\infty}(1-g(v)q^v)^{-1}$ is of the form of

(i) the product $\prod_{p_v}(1-p_v^{-\kappa})^{-1}$ as in (1.2), and

(ii) the product $\prod_{v=1}^{\infty}(1-q^v)^{-1}$ in (1.3) ,

so it creates interest us.

Theorem 3.1. *If the product*

$$\frac{1}{u_{e,\infty}(g;q)}=\prod_{v=1}^{\infty}\left(1-g(2v)q^{2v}\right)^{-1}$$

converges, then there are other equivalent forms of it, viz.

$$(i)\quad \frac{1}{u_{e,\infty}(g;q)}=\sum_{\mu\in P^e}q^{|\mu|}\prod_{\mu_i\in\mu}g(\mu_i)$$

$$(ii)\quad \frac{1}{u_{e,\infty}(g;q)}=1+\Lambda\quad \text{where}\quad \Lambda=\sum_{v=1}^{\infty}\frac{q^{2v}g(2v)}{u_{e,2v}(g;q)}$$

$$(iii)\quad \frac{1}{u_{e,\infty}(g;q)}=1+\frac{T}{u_{e,\infty}(g;q)}\quad \text{where}\quad T=\sum_{v=1}^{\infty}q^{2v}g(2v)u_{e,2v-2}(g;q)$$

$$(iv)\quad \frac{1}{u_{e,\infty}(g;q)}=1+\sum_{v=1}^{\infty}\frac{(-1)^v q^{(-1)v(v-1)}}{u_{e,v}(\frac{1}{g};q^{-1})\prod_{k=1}^{v-1}g(2k)}$$

$$(v)\quad \frac{1}{u_{e,\infty}(g;q)}=1+\cfrac{T}{1-\cfrac{\Lambda}{1+\cfrac{T}{1-\cfrac{\Lambda}{1+\cdots}}}}$$

where P^e be the set of partitions into even distinct parts.

Proof Let P^e be the set of partitions into even distinct parts . Since $l(\emptyset) = 0$, $v_\emptyset = 1$, and considering the even distinct parts of partitions, so we notice that

$$\begin{aligned}&2 = 2, \qquad 4 = 4, \qquad 6 = 6,\ 4+2,\\ &8 = 8,\ 6+2, \qquad 10 = 10,\ 8+2,\ 6+4,\ \cdots\end{aligned}$$

(i) By definition of $u_{e,\infty}(g;q)$, we have

$$\begin{aligned}\frac{1}{u_{e,\infty}(g;q)} =&\prod_{v=1}^{\infty}\left(1-g(2v)q^{2v}\right)^{-1}\\
=&(1-g(2)q^2)^{-1}(1-g(4)q^4)^{-1}(1-g(6)q^6)^{-1}(1-g(8)q^8)^{-1}(1-g(10)q^{10})^{-1}.....\\
=&(1+g(2)q^2+g(2)^2q^{2+2}+g(2)^3q^{2+2+2}+g(2)^4q^{2+2+2+2}+\cdots)\\
&(1+g(4)q^4+g(4)^2q^{4+4}+\cdots)\\
&(1+g(6)q^6+g(6)^2q^{6+6}+\cdots)\\
&(1+g(8)q^8+\cdots)\cdots\\
=&1+g(2)q^2+(g(2)^2q^{2+2}+g(4)q^4)+(g(2)^3q^{2+2+2}+g(2)g(4)q^{2+4}+g(6)q^6)\\
&+(g(2)^4q^{2+2+2+2}+g(2)g(2)g(4)q^{2+2+4}+g(4)^2q^{4+4}+g(2)g(6)q^{2+6}+g(8)q^8)+\cdots\\
=&1+\sum_{v=1}^{\infty}q^{2v}\sum_{\mu\vdash 2v}\prod_{\mu_i\in\mu}g(\mu_i)\\
=&\sum_{\mu\in P^e}q^{|\mu|}\prod_{\mu_i\in\mu}g(\mu_i)\end{aligned}$$

(ii) Next,

$$\begin{aligned}\frac{1}{u_{e,\infty}(g;q)} &= \frac{1}{u_{e,0}(g;q)}+\sum_{v=1}^{\infty}\left[\frac{1}{u_{e,2v}(g;q)}-\frac{1}{u_{e,2v-2}(g;q)}\right]\\
&= 1+\sum_{v=1}^{\infty}\left[\frac{1}{\prod_{v=1}^{\infty}(1-g(2v)q^{2v})}-\frac{1}{\prod_{v=1}^{\infty}(1-g(2v-2)q^{2v-2})}\right]\\
&= 1+\sum_{v=1}^{\infty}\frac{1}{\prod_{v=1}^{\infty}(1-g(2v-2)q^{2v-2})}\left[\frac{1}{1-g(2v)q^{2v}}-1\right]\\
&= 1+\sum_{v=1}^{\infty}\frac{1}{u_{e,2v-2}(g;q)}\left[\frac{1}{1-g(2v)q^{2v}}-1\right]\\
&= 1+\sum_{v=1}^{\infty}\frac{q^{2v}g(2v)}{u_{e,2v}(g;q)}\\
&= 1+\Lambda \quad \text{where} \quad \Lambda=\sum_{v=1}^{\infty}\frac{q^{2v}g(2v)}{u_{e,2v}(g;q)}\end{aligned}$$

(iii) Now,

$$\begin{aligned}
u_{e,\infty}(g;q) &= u_{e,0}(g;q) + \sum_{v=1}^{\infty}\left(u_{e,2v}(g;q) - u_{e,2v-2}(g;q)\right)\\
&= 1 + \sum_{v=1}^{\infty}\left[\prod_{v=1}^{\infty}\left(1-g(2v)q^{2v}\right) - \prod_{v=1}^{\infty}\left(1-g(2v-2)q^{2v-2}\right)\right]\\
&= 1 + \sum_{v=1}^{\infty}\left[\prod_{v=1}^{\infty}\left(1-g(2v-2)q^{2v-2}\right)\left(1-g(2v)q^{2v}-1\right)\right]\\
&= 1 - \sum_{v=1}^{\infty} q^{2v} g(2v)\ u_{e,2v-2}(g;q)
\end{aligned}$$

Hence,

$$u_{e,\infty}(g;q) = 1 - T \quad \text{where} \quad T = \sum_{v=1}^{\infty} q^{2v} g(2v)\ u_{e,2v-2}(g;q). \tag{3.2}$$

From Theorem 3.1 (ii), we have

$$\Lambda = \frac{1}{u_{e,\infty}(g;q)} - 1 = \frac{1-u_{e,\infty}(g;q)}{u_{e,\infty}(g;q)}$$

Using (3.2), we get

$$\Lambda = \frac{T}{u_{e,\infty}(g;q)}. \tag{3.3}$$

Theorem 3.1 (ii) and the above expression give Theorem 3.1 (iii).

(iv) Consider

$$\begin{aligned}
u_{e,v}\left(\frac{1}{g};\frac{1}{q}\right) &= \prod_{k=1}^{v}\left[1-\frac{1}{g(2k)q^{2k}}\right]\\
&= \left(1-\frac{1}{g(2)q^{2}}\right)\left(1-\frac{1}{g(4)q^{4}}\right)\cdots\left(1-\frac{1}{g(2v)q^{2v}}\right)\\
&= \frac{(g(2)q^2-1))\,(g(4)q^4-1)\cdots(g(2v)q^{2v}-1)}{g(2)q^2\ g(4)q^4\ \cdots g(2v)q^{2v}}\\
&= \frac{(-1)^v(1-g(2)q^2)(1-g(4)q^4)\cdots(1-g(2v)q^{2v})}{q^{v(v+1)}\prod_{k=1}^{v} g(2k)}\\
&= \frac{(-1)^v\prod_{k=1}^{v}(1-g(2k)q^{2k})}{q^{v(v+1)}\prod_{k=1}^{v} g(2k)}\\
\therefore\quad u_{e,v}\left(\frac{1}{g};\frac{1}{q}\right) &= \frac{(-1)^v\ u_{e,v}(g;q)}{q^{v(v+1)}\prod_{k=1}^{v} g(2k)}\\
\text{Hence,}\quad \frac{1}{u_{e,v}(g;q)} &= \frac{(-1)^v}{q^{v(v+1)}\ u_{e,v}\left(\frac{1}{g};\frac{1}{q}\right)\prod_{k=1}^{v} g(2k)}
\end{aligned}$$

Using the above equation in Theorem 3.1 (ii), we get

$$\begin{aligned}
\frac{1}{u_{e,\infty}(g;q)} &= 1 + \sum_{v=1}^{\infty}\frac{(-1)^v q^{2v} g(2v)}{q^{v(v+1)}\ u_{e,n}\left(\frac{1}{g};\frac{1}{q}\right)\prod_{k=1}^{v} g(2k)}\\
&= 1 + \sum_{v=1}^{\infty}\frac{(-1)^v q^{-v(v-1)}}{u_{e,v}\left(\frac{1}{g};\frac{1}{q}\right)\prod_{k=1}^{v-1} g(2k)}
\end{aligned}$$

(v) From Theorem 3.1 (ii) and Theorem 3.1 (iii), we have

$$\begin{aligned}\frac{1}{u_{e,\infty}(g;q)} &= 1+\frac{T}{u_{e,\infty}(g;q)}\\ &= 1+\frac{T}{1-u_{e,\infty}(g;q)\ \Lambda}\\ &= 1+\frac{T}{1-\frac{\Lambda}{\frac{1}{u_{e,\infty}(g;q)}}}\\ &= 1+\frac{T}{1-\frac{\Lambda}{1+\frac{T}{1-\frac{\Lambda}{1+\cdots}}}}\end{aligned}$$

Theorem 3.2. *If the product* $u_{e,\infty}(g;q)=\prod_{v=1}^{\infty}\left(1-g(2v)q^{2v}\right)$ *converges, then there are other equivalent forms of it, viz.*

$$\begin{aligned}&(i) \quad u_{e,\infty}(g;q) = \sum_{\mu\in P^e}(-1)^{l(\mu)}q^{|\mu|}\prod_{\mu_i\in\mu}g(\mu_i)\\ &(ii) \quad u_{e,\infty}(g;q) = 1-T\\ &(iii) \quad u_{e,\infty}(g;q) = 1-u_{e,\infty}(g;q)\ \Lambda\\ &(iv) \quad u_{e,\infty}(g;q) = 1-\frac{\Lambda}{1+\frac{T}{1-\frac{\Lambda}{1+\cdot}}}\end{aligned}$$

where P^e, Λ *and* T *are exactly as in Theorem 3.1.*

Proof *(i)* Consider

$$\begin{aligned}&\sum_{\mu\in P^e}(-1)^{l(\mu)}q^{|\mu|}\prod_{\mu_i\in\mu}g(\mu_i)\\ &=1+(-1)q^2g(2)+(-1)q^4g(4)\\ &+(-1)q^6g(6)+q^6f(4)g(2)\\ &+(-1)q^8g(8)+q^8g(6)f(2)\\ &+(-1)q^{10}g(10)+q^{10}g(8)g(2)+q^{10}g(6)g(4)+\cdots\end{aligned} \tag{3.4}$$

$$\begin{aligned}\text{and}\quad u_{e,\infty}(g;q) &= \prod_{v=1}^{\infty}\left(1-g(2v)q^{2v}\right)\\ &=\left(1-g(2)q^2\right)\cdot\left(1-g(4)q^4\right)\cdot\left(1-g(6)q^6\right)\\ &\quad\cdot\left(1-g(8)q^8\right)\cdot\left(1-g(10)q^{10}\right)\cdots\\ &=1+(-1)q^2g(2)+(-1)q^4g(4)+(-1)q^6g(6)+q^6g(4)g(2)\\ &\quad+(-1)q^8g(8)+q^8g(6)g(2)+(-1)q^{10}g(10)\\ &\quad+q^{10}g(8)g(2)+q^{10}g(6)g(4)+\cdots\end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we obtain Theorem 3.2 (i).

$$
\begin{aligned}
(ii) \qquad u_{e,\infty}(g;q) &= u_{e,0}(g;q) + \sum_{v=1}^{\infty} \left(u_{e,2v}(g;q) - u_{e,2v-2}(g;q)\right) \\
&= 1 + \sum_{v=1}^{\infty} \left[\prod_{v=1}^{\infty} \left(1 - g(2v)q^{2v}\right) - \prod_{v=1}^{\infty} \left(1 - g(2v-2)q^{2v-2}\right) \right] \\
&= 1 + \sum_{v=1}^{\infty} \left[\prod_{v=1}^{\infty} \left(1 - g(2v-2)q^{2v-2}\right) \left(1 - g(2v)q^{2v} - 1\right) \right] \\
&= 1 - \sum_{v=1}^{\infty} q^{2v} g(2v)\ u_{e,2v-2}(g;q) \\
&= 1 - T \quad \text{where } T = \sum_{v=1}^{\infty} q^{2v} g(2v)\ u_{e,2v-2}(g;q)
\end{aligned}
$$

(iii) From Theorem 3.1 (ii), we have

$$
\begin{aligned}
\frac{1}{u_{e,\infty}(g;q)} &= 1 + \Lambda \quad \text{where} \quad \Lambda = \sum_{v=1}^{\infty} \frac{q^{2v} g(2v)}{u_{e,2v}(g;q)} \\
\implies u_{e,\infty}(g;q) &= 1 - u_{e,\infty}(g;q)\ \Lambda
\end{aligned}
$$

(iv) From the above result, we have

$$
\begin{aligned}
u_{e,\infty}(g;q) &= 1 - u_{e,\infty}(g;q)\ \Lambda \\
&= 1 - \frac{\Lambda}{\frac{1}{u_{e,\infty}(g;q)}} \\
&= 1 - \frac{\Lambda}{1 + \frac{T}{1 - \frac{\Lambda}{1 + \cdot}}}
\end{aligned}
$$

where Λ, T are exactly as in Theorem 3.1.

4. SUMS OVER PARTITIONS INTO ODD DISTINCT PARTS, 1 AS ONLY PART, AND $v = v$ AS PART

In this section we prove the theorems taking sum over partitions as odd distinct parts, 1 as only part, and $v = v$ as part.

Theorem 4.1. *If the product* $u_{o,\infty}(g;q) = \prod_{v=1}^{\infty}(1 - g(2v-1)q^{2v-1})$ *converges, then it can be expressed in the following equivalent form*

$$
u_{o,\infty}(g;q) = \sum_{\mu \in P^o} (-1)^{l(\mu)} q^{|\mu|} \prod_{\mu_i \in \mu} g(\mu_i) \tag{4.1}
$$

where P^o *is the set of partitions into odd distinct parts.*

Proof Let P^o be the set of partitions into odd distinct parts. Since $l(\emptyset) = 0$, $n_{\emptyset} = 1$ and considering the odd distinct parts of partitions, we observe that

$$
\begin{aligned}
&1 = 1, \quad 3 = 3, \quad 4 = 3 + 1, \quad 5 = 5, \quad 6 = 5 + 1, \\
&7 = 7, \quad 8 = 5 + 3,\ 7 + 1, \quad 9 = 9,\ 5 + 3 + 1,\ \cdots
\end{aligned}
$$

Consider

$$\begin{aligned}\sum_{\mu\in P^o}(-1)^{l(\mu)}q^{|\mu|}\prod_{\mu_i\in\mu}g(\mu_i) =&1+(-1)q^1g(1)+(-1)q^3g(3)+(-1)^2q^4g(1)g(3)\\ &+(-1)q^5g(5)+(-1)^2q^6g(5)g(1)+(-1)q^7g(7)\\ &+(-1)^2q^8g(5)g(3)+(-1)^2q^8g(7)g(1)\\ &+(-1)q^9g(9)+(-1)^3q^9g(5)g(3)g(1)+\ \cdots\\ =&1-q^1f(1)-q^3g(3)+q^4g(1)g(3)\\ &-q^5g(5)+q^6g(5)g(1)-q^7g(7)\\ &+q^8\left[g(5)g(3)+g(7)g(1)\right]\\ &-q^9\left[g(9)+g(5)g(3)g(1)\right]+\ \cdots\end{aligned}\tag{4.2}$$

and

$$\begin{aligned}u_{o,\infty}(g;q)=&\prod_{v=1}^{\infty}(1-g(2v-1)q^{2v-1})\\ =&(1-g(1)q^1)(1-g(3)q^3)(1-g(5)q^5)\\ &(1-g(7)q^7)(1-g(9)q^9)\cdots\\ =&1-q^1g(1)-q^3g(3)+q^4g(1)g(3)-q^5f(5)+\\ &q^6g(5)g(1)-q^7g(7)+q^8\left[g(5)g(3)+(g(7)g(1)\right]-\\ &q^9\left[g(9)+g(5)g(3)g(1)\right]\cdots\end{aligned}\tag{4.3}$$

From (4.2) and (4.3), we obtain (4.1).

Theorem 4.2. *If P^1 be the set of partitions such that all parts are 1 only, then*

$$\sum_{\mu\in P^1}(-1)^{l(\mu)}q^{|\mu|}\prod_{\mu_i\in\mu}g(\mu_i)=(1+qg(1))^{-1}\quad \textit{where}\quad |qg(1)|<1.$$

If $g(1)=1$, we get

$$\sum_{\mu\in P^1}(-1)^{l(\mu)}q^{|\mu|}=(1+q)^{-1}$$

Proof Conventionally, $l(\emptyset)=0,\ |\emptyset|=0.$ Since

$$\begin{aligned}&1=1,\\ &2=1+1,\\ &3=1+1+1,\\ &4=1+1+1+1,\ \cdots,\\ \text{so}\quad &l(1)=1,\ l(2)=2,\ l(3)=3,\ l(4)=4,\ \cdots\\ \text{and}\quad &|1|=1,\ |2|=2,\ |3|=3,\ |4|=4,\ \cdots\end{aligned}$$

Now,

$$\begin{aligned}\sum_{\mu\in P^1}(-1)^{l(\mu)}q^{|\mu|}\prod_{\mu_i\in\mu}g(\mu_i)&=1+(-1)qg(1)+(-1)^2q^2g(1)g(1)\\ &+(-1)^3q^3g(1)g(1)g(1)+\cdots\\ &=1-q\ g(1)+q^2\ \ (g(1))^2-q^3\ \ (g(1))^3+\cdots\\ &=(1+qg(1))^{-1}\quad \text{where}\quad |qg(1)|<1.\end{aligned}$$

Taking $g(1)=1,$ we get

$$\sum_{\mu\in P^1}(-1)^{l(\mu)}q^{|\mu|}=(1+q)^{-1}.$$

Theorem 4.3. *If P^v be the set of partitions such that $v = v$, then*

$$\sum_{\mu \in P^v} (-1)^{l(\mu)} q^{|\mu|} \prod_{\mu_i \in \mu} g(\mu_i) = 1 - \sum_{v \geq 1} q^v g(v).$$

If $g(v) = -1$, then

$$\sum_{\mu \in P^v} (-1)^{l(\mu)} q^{|\mu|} \prod_{\mu_i \in \mu} g(\mu_i) = (1-q)^{-1}.$$

Proof Here $l(\mu) = 1$ for all $v \in \mathbb{N}$ and $|\mu| = 1, 2, 3, \cdots$

$$\sum_{\mu \in P^v} (-1)^{l(\mu)} q^{|\mu|} \prod_{\mu_i \in \mu} g(\mu_i) = 1 - [qg(1) + q^2 g(2) + q^3 g(3) + ...]$$

$$= 1 - \sum_{v \geq 1} q^v gv).$$

If $g(v) = -1$, then

$$\sum_{\mu \in P^v} (-1)^{l(\mu)} q^{|\mu|} \prod_{\mu_i \in \mu} g(\mu_i) = (1-q)^{-1}.$$

5. CONCLUSION

Here we presented interesting results for the sum over partitions as even distinct parts, odd distinct parts, 1 as only part and $v = v$ as part. At last, we conclude that if one consider the sums of the form $\sum_\mu g(\mu)$ over different partitions, then some more patterns may be obtained.

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