Approximation of Functions By Means of Conjugate Fourier Series in Lp- Norm

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Abstract— In this paper, we estimate the degree of approximation of $\tilde{f}$, conjugate of $f \in Lip(\omega(t), p)$ - class by matrix means of conjugate Fourier series of f in terms of modulus of continuity. We also discuss some results which are analogous to our results.

Keywords — Degree of approximation, $Lip(\omega(t), p)$ – class, conjugate Fourier series, Modolus of continuity.

I. INTRODUCTION

Let f be a 2π periodic function belonging to $L^p := L^p[0,2\pi](p \geq 1)$ -space. The trigonometric Fourier series of f is defined as

$$f(x) \sim a_0/2 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx).$$

The k^{th} partial sums of the Fourier series of f, that is,

$$s_k(f;x) := \frac{a_0}{2} + \sum_{\nu=1}^{k}(a_\nu \cos \nu x + b_\nu \sin \nu x), \quad k \in \mathbb{N} \qquad (1)$$ with $s_0(f;x) = \frac{a_0}{2}$, called trigonometric polynomial of degree or (order) k.

The conjugate series of the Fourier series of f is defined by

$$\sum_{k=1}^{\infty}(a_k \sin kx - b_k \cos kx),$$

with its k^{th} partial sums

$$\tilde{s}_k(f;x) := \sum_{\nu=1}^{k}(a_\nu \sin \nu x - b_\nu \cos \nu x), \; k \in \mathbb{N} \qquad (2)$$ and $\tilde{s}_0(f;x) = 0.$

The conjugate of f denoted by $\tilde{f}$ is defined as

$$\tilde{f}(x) = -\frac{1}{2\pi} \lim_{\epsilon \to 0} \int_{\epsilon}^{\pi} \psi(x,t) \cot(t/2)\, dt, \qquad (3)$$ where $\psi(x,t) = f(x+t) - f(x-t)$ [6].

The L^p norm of $f \in L^p[0,2\pi]$ is defined by

$$\|f\|_p = \begin{cases} \left(\dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} |f(x)|^p \, dx \right)^{1/p}, & 1 \le p < \infty \\ \operatorname{ess} \sup\limits_{x \in [0,2\pi]} |f(x)|, & p = \infty. \end{cases} \tag{4}$$

The degree of approximation $E_n(f)$ of a function $f \in L^p$ -space by a trigonometric polynomial $T_n(x)$ of degree $\le n$ is given by

$$E_n(f) = \min_{T_n} \|f(x) - T_n(x)\|_p .$$

This method of approximation is called the trigonometric Fourier approximation.

Recently, Srivastava and Singh [5] have defined a function class:

$Lip(\omega(t), p) = \left\{ f \in L^p[0,2\pi] : \|f(x+t) - f(x)\|_p = O(t^{-1/p}\omega(t)) \right\}$, where $t > 0, p \ge 1$ and $\omega(t)$ is a positive increasing function of t, which generalize the definition of $Lip(\xi(t), p)$ defined by Khan and Ram [1, p.47] and classical Lipschitz classes $Lip(\xi(t), p)$, $Lip(\alpha, p)$, and $Lip\alpha$ [5, p.224].

Define

$$t_n(f;x) := \sum_{k=0}^{n} a_{n,k} s_k(f;x), \; n \in \mathbb{N}_0$$

where $T = (a_{n,k})$ be a lower triangular matrix with non-negative entries such that $a_{n,-1} = 0, A_{n,k} = \sum_{r=k}^{n} a_{n,r}$ and $A_{n,0} = 1, \; \forall n \in \mathbb{N}_0$. The Fourier series of a function f is said to be $T-$summable to s, if $t_n(f;x) \to s$ as $n \to \infty$.

We write $\tilde{K}_n(t) = \dfrac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} \dfrac{\cos(n-k+1/2)t}{\sin(t/2)}$, $\tau = \left[\dfrac{1}{t} \right]$, the integer part of $\dfrac{1}{t}$ and $\tilde{t}_n(f;x) = \sum_{k=0}^{n} a_{n,k} \tilde{s}_k(f;x), n \in \mathbb{N}_0$.

II. KNOWN RESULTS

Recently Srivastava and Singh [5] have proved the following theorem:

Theorem A [5, Theorem 1]: *Let* $T \equiv (a_{n,k})$ *be a lower triangular matrix with non-negative and non-decreasing* (with respect to k) *entries. Then the degree of approximation of a* 2π *-periodic function* $f \in Lip(\omega(t), p)$ *with* $p \ge 1$ *by matrix means of its Fourier series is given by*

$$\|t_n(f;x) - f(x)\|_p = O\left((n+1)^{1/p} \omega(\pi/(n+1)) \right), \tag{5}$$

providing $\omega(t)$ *is a positive increasing function and satisfies the following conditions:*

$$\frac{\omega(t)}{t^{\sigma}} \text{ is an increasing function for some } 0 < \sigma < 1, \tag{6}$$

$$\frac{|\phi(t)|}{\omega(t)\, t^{-1/p}} \text{ is bounded funcion of } t, \tag{7}$$

$$\left(\int_{\pi/(n+1)}^{\pi}\left(\frac{\omega(t)}{t^{1+1/p}}\right)^{p}dt\right)^{1/p}=O\left((n+1)\,\omega\left(\frac{\pi}{n+1}\right)\right),\qquad (8)\ \textit{where } p^{-1}+q^{-1}=1.\ \textit{Also condition (7) holds uniformly in } x.$$

Further, Lal and Mishra [2] and, very recently, Rhoades [4] have proved their results in certain Lipschitz classes. From these results, we observe that the conditions (6)-(8) can be replace by a single condition. This has motivated us to study Theorem A further for conjugate functions.

III. MAIN RESULTS

In this paper, we shall prove conjugate version of the theorem of [5] by relaxing conditions like (6)-(8).

Theorem 1: *Let* f *be a* 2π *-periodic function belonging to* $Lip(\omega(t),p)$ *-class with* $p\ge 1$ *and let* $T\equiv(a_{n,k})$ *be a lower triangular regular matrix with non-negative and non-decreasing* (with respect to $0\le k\le n$) *entries with* $A_{n,0}=1$. *Then the degree of approximation of* $\tilde{f}$, *conjugate of* f, *by matrix means of conjugate Fourier series is given by*

$$\left\|\tilde{t}_n(f;x)-\tilde{f}(x)\right\|_p=O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\omega(t)}{t^{2+1/p}}dt\right),\qquad (9)$$

provided $\omega(t)$ is a positive increasing function satisfying the following condition:

$$\int_0^v\frac{\omega(t)}{t^{1+1/p}}dt=O\left(\frac{\omega(v)}{v^{1/p}}\right),\ 0<v<\pi.\qquad (10)$$

Remark 1: *For* $\omega(t)=t^{1/p}\xi(t)$, *the* $Lip(\omega(t),p)-$*class coincides with* $Lip(\xi(t),p)-$*class and condition* (10) *reduces to condition* (6) *of Rhoades* [4, *Theorem 5, p.* 395]. *Thus our theorem extends Theorem* 5 *and* 6 *of Rhoades* [4] *and Theorems* 3.1, 3.2 *of Lal and Mishra* [2] *to their matrix analogous.*

IV. LEEMAS

We need the following lemmas for proving our theorem**.**

Lemma 1: *Let* $T\equiv(a_{n,k})$ *be a lower triangular regular matrix with non-negative and non-decreasing* (with respect to $0\le k\le n$) *entries with* $A_{n,0}=1$. *Then*

$$|\tilde{K}_n(t)|=\begin{cases}O\left(\dfrac{1}{t}\right), & \text{for } 0<t\le 1/(n+1)\\[2mm] O\left(\dfrac{1}{(n+1)t^2}\right), & \text{for } 1/(n+1)<t\le\pi.\end{cases}$$

Proof: Case I: For $0<t\le 1/(n+1)$, using $\dfrac{1}{\sin(t/2)}=O\left(\dfrac{\pi}{t}\right)$, we have

$$\begin{aligned}|\tilde{K}_n(t)| &=\left|\frac{1}{2\pi}\sum_{k=0}^{n}a_{n,n-k}\frac{\cos(n-k+1/2)t}{\sin(t/2)}\right|\\ &\le\frac{1}{2\pi}\sum_{k=0}^{n}a_{n,n-k}\left|\frac{\cos(n-k+1/2)t}{\sin t/2}\right|\\ &=O\left(\frac{1}{t}\right)\sum_{k=0}^{n}a_{n,n-k}=O\left(\frac{1}{t}\right)A_{n,0}=O\left(\frac{1}{t}\right),\qquad (11)\end{aligned}$$

in view of $A_{n,0}=1$.

Case II: For $1/(n+1)<t\le\pi,$ using $\dfrac{1}{\sin(t/2)}=O\left(\dfrac{\pi}{t}\right)$ and $\tau=\left[\dfrac{1}{t}\right],$ we have

$$|\tilde{K}_n(t)|=\left|\frac{1}{2\pi}\sum_{k=0}^{n}a_{n,n-k}\,\frac{\cos(n-k+1/2)t}{\sin(t/2)}\right|$$

$$=O\left(\frac{1}{t}\right)\left|\sum_{k=0}^{n}a_{n,n-k}\cos(n-k+1/2)t\right|$$

$$=O\left(\frac{1}{t}\right)\left|\mathrm{Re}\left(\sum_{k=0}^{n}a_{n,n-k}e^{i(n-k+1/2)t}\right)\right|$$

$$=O\left(\frac{1}{t}\right)|\sum_{k=0}^{n}a_{n,n-k}e^{i(n-k)t}|.$$

Following McFadden [3, Lemma 5.11, p.8], we have $\left|\sum_{k=0}^{n}a_{n,n-k}e^{i(n-k)t}\right|=\left|e^{int}\sum_{k=0}^{n}a_{n,n-k}e^{-ikt}\right|$

$$\le|\sum_{k=0}^{\tau-1}a_{n,n-k}e^{-ikt}|+|\sum_{k=\tau}^{n}a_{n,n-k}e^{-ikt}|$$

$$\le\sum_{k=0}^{\tau-1}a_{n,n-k}+2a_{n,n-\tau}\max_{\tau\le k\le n}\left|\frac{1-e^{-i(k+1)t}}{1-e^{-it}}\right|$$

$$\le A_{n,n-\tau+1}+2a_{n,n-\tau}(1/\sin(t/2))$$

$$\le A_{n,n-\tau}+2(\tau+1)a_{n,n-\tau}=O(A_{n,n-\tau}), \qquad (12)$$

in view of increasing nature of $a_{n,k}$ i.e., $2(\tau+1)a_{n,n-\tau}=O(A_{n,n-\tau}).$

Further, using the regularity conditions of $T,$ we have

$$A_{n,n-\tau}=O\left(\frac{1}{t(n+1)}\right) \qquad (13)$$

Hence,

$$|\tilde{K}_n(t))|=O\left(\frac{1}{t^2}(n+1)\right). \qquad (14)$$

Collecting (11) - (14), Lemma 1 is completed.

Lemma 2: [6]. Let $g(x,t)\in L^p([a,b]\times[c,d]),$ for $p\ge1.$ Then

$$\left\{\int_a^b\left|\int_c^d g(x,t)dt\right|^p dx\right\}^{1/p}\le\int_c^d\left\{\int_a^b|g(x,t)|^p dx\right\}^{1/p}dt$$

which is known as generalized form of Minkowski's inequality.

V. PROOF OF MAIN RESULTS

Proof of Theorem 1:

The integral representation of $\tilde{s}_n(f;x)$ defined in (2) is given by

$$\tilde{s}_n(f;x) = -\frac{1}{\pi}\int_0^{\pi}\psi(x,t)\left\{\frac{\cos(t/2)-\cos(n+1/2)t}{2\sin(t/2)}\right\}dt$$ so that

$$\tilde{s}_n(f;x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^{\pi}\psi(x,t)\frac{\cos(n+1/2)t}{\sin(t/2)}dt$$ and

$$\begin{aligned}\tilde{t}_n(f;x) - \tilde{f}(x) &= \frac{1}{2\pi}\int_0^{\pi}\psi(x,t)\sum_{k=0}^{n}a_{n,n-k}\frac{\cos(n-k+1/2)t}{\sin(t/2)}dt\\ &= \int_0^{\pi}\psi(x,t)\frac{1}{2\pi}\sum_{k=0}^{n}a_{n,n-k}\frac{\cos(n-k+1/2)t}{\sin(t/2)}dt\\ &= \int_0^{\pi}\psi(x,t)\tilde{K}_n(t)dt.\end{aligned}$$

Using Lemma 2 for $0 \le t \le \pi$, we have

$$\begin{aligned}\left\|\tilde{t}_n(f;x) - \tilde{f}(x)\right\|_p &= \left(\frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^{\pi}\psi(x,t)\tilde{K}_n(t)dt\right|^p dx\right)^{1/p}\\ &\le \int_0^{\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}|\psi(x,t)|^p dx\right)^{1/p}|\tilde{K}_n(t)|\,dt\\ &= \int_0^{\pi}\frac{\omega(t)}{t^{1/p}}|\tilde{K}_n(t)|\,dt\\ &= \left(\int_0^{1/(n+1)} + \int_{1/(n+1)}^{\pi}\right)\left(\frac{\omega(t)}{t^{1/p}}|\tilde{K}_n(t)|\,dt\right)\\ &= I_1 + I_2. \qquad (15)\end{aligned}$$

Now, using Lemma 1 for $0 < t \le 1/(n+1)$, we have

$$\begin{aligned}I_1 &= \int_0^{1/(n+1)}\frac{\omega(t)}{t^{1/p}}|\tilde{K}_n(t)|\,dt\\ &= O\left(\int_0^{1/(n+1)}\frac{\omega(t)}{t^{1+1/p}}dt\right) = O\left(\omega\left(\frac{1}{n+1}\right)(n+1)^{1/p}\right), \qquad (16)\end{aligned}$$

in view of condition (10).

Again, using Lemma 1 for $1/(n+1) < t \le \pi$, we have

$$\begin{aligned}I_2 &= \int_{1/(n+1)}^{\pi}\frac{\omega(t)}{t^{1/p}}|\tilde{K}_n(t)|\,dt\\ &= O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\omega(t)}{t^{2+1/p}}dt\right).\end{aligned}$$

Now,

$$\begin{aligned}\frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\omega(t)}{t^{2+1/p}}dt &\ge \frac{1}{(n+1)}\omega\left(\frac{1}{n+1}\right)\int_{1/(n+1)}^{\pi}\frac{1}{t^{2+1/p}}dt\\ &= \frac{1}{n+1}\omega\left(\frac{1}{n+1}\right)\left[(n+1)^{1+1/p} - \frac{1}{\pi^{1+1/p}}\right]\end{aligned}$$

$$= \frac{1}{n+1}\omega\left(\frac{1}{n+1}\right)(n+1)^{1+1/p}\left[1-\frac{1}{\left((n+1)\pi\right)^{1+1/p}}\right]$$

$$\geq \frac{1}{2}\omega\left(\frac{1}{n+1}\right)(n+1)^{1/p},$$

that is

$$\omega\left(\frac{1}{n+1}\right)(n+1)^{1/p} = O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\omega(t)}{t^{2+1/p}}dt\right). \qquad (17)$$

Collecting (15)-(17), we get

$$\left\|\tilde{t}_n(f;x)-\tilde{f}(x)\right\|_p = O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\omega(t)}{t^{2+1/p}}dt\right).$$

Thus the proof of Theorem 1 is completed.

Remark 2: *The Theorem* A *can also be proved by using the method of Theorem* 1**.**

VI. COROLLARIES

1. If $\omega(t)=t^{1/p}\xi(t)$, then $f\in Lip(\xi(t),p)$-class and condition (10) reduces to the condition

$$\int_0^v\frac{\xi(t)}{t}dt = O(\xi(v)), 0<v<\pi.$$

Thus for $f\in Lip(\xi(t),p)$, we have

$$\left\|\tilde{t}_n(f;x)-\tilde{f}(x)\right\|_p = O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\xi(t)}{t^2}dt\right).$$

2. If $\omega(t)=t^{\alpha+1/p}$, then $f\in Lip(\alpha,p)$-class and condition (10) reduces to the condition

$$\int_0^v t^{\alpha-1}dt = O(v^{\alpha}), 0<v<\pi.$$

Thus for $f\in Lip(\alpha,p)$, we have

$$\left\|\tilde{t}_n(f;x)-\tilde{f}(x)\right\|_p = O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi}t^{\alpha-2}dt\right)$$

$$= \begin{cases} O\left((n+1)^{-\alpha}\right), & \text{for } 0<\alpha<1 \\ O\left(\dfrac{\log\pi(n+1)}{n+1}\right), & \text{for } \alpha=1. \end{cases}$$

3. Corollaries 1 and 2 are analogous to the results given by Rhoades [4, Theorem 5, Theorem 6, p.395] and Lal and Mishra [2, Theorem 3.1, Theorem 3.2, pp. 4-5].

REFERENCES

[1] Khan, Huzoor H.; Ram, Govind. On the degree of approximation. Facta Univ. Ser. Math. Inform. No. 18 (2003), 47-57.

[2] Lal, Shyam; Mishra, Abhishek. Euler-Hausdorff matrix summability operator and trigonometric approximation of the conjugate of a function belonging to the generalized Lipschitz class. J. Inequal. Appl. 2013, 2013:59, 14 pp.

[3] McFadden, Leonard. Absolute Norlund summability. Duke Math. J. 9, (1942). 168-207.

[4] Rhoades, B. E. The degree of approximation of functions, and their conjugates, belonging to several general Lipschitz classes by Hausdorff matrix means of the Fourier series and conjugate series of a Fourier series. Tamkang Journal of Mathematics 45 (2014), 389-395.

[5] Srivastava, Shailesh Kumar; Singh, Uaday. Trigonometric approximation of periodic functions belonging to $Lip(\omega(t), p)$ -class. J. Comput. Appl. Math. 270 (2014), 223-230.

[6] Zygmund, A., Trigonometric series, Cambridge Univ. Press, Cambridge (III Edition, 2002).