On a Subclass of Harmonic Univalent Functions Defined by Using a New Differential Operator

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Abstract In this paper, a class of complex-valued harmonic univalent functions $f(z) = h(z) + g(\bar{z})$ in the open disk $U = \{z : z \in Cand|z| < 1\}$ is defined by using a new differential operator. We investigate coefficient bounds, distortion inequalities, extreme points and convex combination results for this class.

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I Introduction

Harmonic functions are famous for their use in the study of minimal surfaces and also play important roles in a variety of problems in applied mathematics for this see [3], [5], [6]. A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain C if both u and v are real harmonic in C. In any simply connected domain $D \subseteq C$ we can write $f(z) = h(z) + \overline{g(z)}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense- preserving in D is that $\left|h^{'}(z)\right| > \left|g^{'}(z)\right|$ in D; for this see [4].

Denote by S_H the class of functions $f(z) = h(z) + \overline{g(z)}$ that are harmonic univalent and sense-preserving in the unit disk $U = \{z : z \in Cand|z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f(z) = h(z) + \overline{g(z)} \in S_H$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad z \in U. \tag{1}$$

Therefore

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}, \quad |b_1| < 1.$$

Note that S_H reduces to the class S normalized analytic univalent functions in U if the co-analytic part of f is identically zero.

In [4] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on H and its subclasses such as [1], [10], [11], [7] and [9] studied the harmonic univalent functions.

The differential operator $D^n_{\alpha,\mu}(\lambda, w)(n \in N)$ was introduced in [2]. For $f(z) = h(z) + g(\bar{z})$ given by (1), [12] defined the differential operator as

$$D^n_{\alpha,\mu}(\lambda, w)f(z) = D^n_{\alpha,\mu}(\lambda, w)h(z) + (-1)^n \overline{D^n_{\alpha,\mu}(\lambda, w)g(z)}$$

where

$$D^n_{\alpha,\mu}(\lambda, w)h(z) = z + \sum_{k=2}^{\infty} \left[(k-1)\left(\mu w^{\lambda} - \alpha\right) + k\right]^n a_k z^k$$

and

$$D^n_{\alpha,\mu}(\lambda, w)g(z) = \sum_{k=1}^{\infty} \left[(k+1)\left(\mu w^{\lambda} - \alpha\right) + k\right]^n b_k z^k,$$

where, $\mu, \lambda, w \geq 0, 0 \leq \alpha \leq \mu w^{\lambda}, with D^n_{\alpha,\mu}(\lambda, w)f(0) = 0.$

The generalization of the differential operator for a function $f(z) = h(z) + g(\bar{z})$ given by (1).

$$D^0_{\alpha,\mu}(\lambda, w)f(z) = D^0 f(z) = h(z) + g(z),$$

$$D^1_{\alpha,\mu}(\lambda, w)f(z) = \left(\alpha - \mu w^{\lambda}\right)\left(h(z) + \overline{g(z)}\right) + \left(\mu w^{\lambda} - \alpha + 1\right)\left(zh'(z) - \overline{zg'(z)}\right),$$

In general,

$$D^n_{\alpha,\mu}(\lambda, w)f(z) = D\left(D^{n-1}_{\alpha,\mu}(\lambda, w)f(z)\right). \tag{2}$$

If f is given by (1), then from (2), we see that

$$\begin{aligned} D^n_{\alpha,\mu}(\lambda, w)f(z) = z + \sum_{k=2}^{\infty} \left[(k-1)\left(\mu w^{\lambda} - \alpha\right) + k\right]^n a_k z^k + \\ (-1)^n \sum_{k=1}^{\infty} \left[(k+1)\left(\mu w^{\lambda} - \alpha\right) + k\right]^n \overline{b_k z^k}. \end{aligned} \tag{3}$$

When, $w = \alpha = 0$, we get modified Salgean differential operator [8].

We define $S_H(\lambda, w, n, \alpha, \beta)$ the subclass of S_H consisting of functions f of the form (1) that satisfy the condition

$$Re\left\{\left(1 + e^{it}\right)\frac{D^{n+1}_{\alpha,\mu}(\lambda, w)f(z)}{D^n_{\alpha,\mu}(\lambda, w)f(z)} - e^{it}\right\} \geq \beta, \tag{4}$$

where, $D^n_{\alpha,\mu}(\lambda, w)f(z)$ is defined by (3) $\quad and \quad$ $(0 \leq \beta < 1, t \in R, n, \alpha \in N_0)$.

We let the subclass $\overline{S_H}(\lambda, w, n, \alpha, \beta)$ consisting of harmonic functions $f_n = h + \overline{g_n}$ in S_H so that h and g_n are of the form.

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g_n(z) = (-1)^n \sum_{k=1}^{\infty} b_k z^k, \quad a_k, b_k \geq 0 \tag{5}$$

The objective of the present paper is to give sufficient condition for functions $f = h + \overline{g}$ where h and g are given by (1) to be in the class $S_H(\lambda, w, n, \alpha, \beta)$ and it is shown that this coefficient condition is also necessary for functions belonging to the subclass $\overline{S_H}(\lambda, w, n, \alpha, \beta)$. Also, we obtain coefficient bounds, distortion inequalities, extreme points and convex combination results for this class.

II Coefficient Bounds

In our first theorem, we introduce a sufficient coefficient bound for harmonic functions in $S_H(\lambda, w, n, \alpha, \beta)$.

Theorem II.1 *Let $f = h + \bar{g}$ be so that h and g are of the form (1) . Furthermore, let*

$$\begin{aligned}&\sum_{k=2}^{\infty} \left\{ \left[2\left((k-1)\left(\mu w^{\lambda} - \alpha\right) + k \right) - (1+\beta) \right] \left[(k-1)\left(\mu w^{\lambda} - \alpha\right) + k \right]^n \right\} |a_k| \\ &+ \sum_{k=1}^{\infty} \left\{ \left[2\left((k+1)\left(\mu w^{\lambda} - \alpha\right) + k \right) + (1+\beta) \right] \left[(k+1)\left(\mu w^{\lambda} - \alpha\right) + k \right]^n \right\} |b_k| \\ &\leq (1-\beta),\end{aligned} \tag{6}$$

where $\mu, \lambda, w \geq 0, 0 \leq \alpha \leq \mu w^{\lambda}$, $t \in R$, $n \in N_0$, $0 \leq \beta < 1$. Then f is sense-preserving, harmonic univalent in U and $f \in S_H(\lambda, w, n, \alpha, \beta)$.

Proof: If $z_1 \neq z_2$,

$$\begin{aligned}\left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| &\geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| \\ &= 1 - \left| \frac{\sum_{k=1}^{\infty} b_k \left(z_1^k - z_2^k \right)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k \left(z_1^k - z_2^k \right)} \right| \\ &> 1 - \frac{\sum_{k=1}^{\infty} k\,|b_k|}{1 - \sum_{k=2}^{\infty} k\,|a_k|} \\ &\geq 1 - \frac{\sum_{k=1}^{\infty} \frac{\left[2\left((k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right)+(1+\beta)\right]\left[(k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^n}{1-\beta} |b_k|}{1 - \sum_{k=2}^{\infty} \frac{\left[2\left((k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right)-(1+\beta)\right]\left[(k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^n}{1-\beta} |a_k|} \\ &\geq 0.\end{aligned}$$

Which proves univalence. Note that f is sense preserving in U. This is because

$$\begin{aligned}
\left|h^{'}(z)\right| &\geq 1-\sum_{k=2}^{\infty} k\left|a_k\right|\left|z\right|^{k-1} \\
&> 1-\sum_{k=2}^{\infty} \frac{\left[2\left((k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right)-(1+\beta)\right]\left[(k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n}}{1-\beta}\left|a_k\right| \\
&\geq \sum_{k=1}^{\infty} \frac{\left[2\left((k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right)+(1+\beta)\right]\left[(k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n}}{1-\beta}\left|b_k\right| \\
&> \sum_{k=1}^{\infty} k\left|b_k\right|\left|z\right|^{k-1} \\
&\geq \left|g^{'}(z)\right|.
\end{aligned}$$

Using the fact that $R(w) \geq \beta$ if and only if $|1-\beta+w| \geq |1+\beta-w|$, it suffucient to show that

$$\begin{aligned}
&\left|(1-\beta-e^{it})D^{n}_{\alpha,\mu}(\lambda,w)f(z)+(1+e^{it})D^{n+1}_{\alpha,\mu}(\lambda,w)f(z)\right|- \\
&\left|(1+\beta+e^{it})D^{n}_{\alpha,\mu}(\lambda,w)f(z)-(1+e^{it})D^{n+1}_{\alpha,\mu}(\lambda,w)f(z)\right| \geq 0
\end{aligned} \tag{7}$$

Substituting for $D^{n+1}_{\alpha,\mu}(\lambda,w)f(z)$ and $D^{n}_{\alpha,\mu}(\lambda,w)f(z)$ in (7), we obtain

$$\begin{aligned}
&\left|(1-\beta-e^{it})D^{n}_{\alpha,\mu}(\lambda,w)f(z)+(1+e^{it})D^{n+1}_{\alpha,\mu}(\lambda,w)f(z)\right| \\
&-\left|(1+\beta+e^{it})D^{n}_{\alpha,\mu}(\lambda,w)f(z)-(1+e^{it})D^{n+1}_{\alpha,\mu}(\lambda,w)f(z)\right| \\
&= \left|\begin{array}{l}
\left(1-\beta-e^{it}\right)\left\{\begin{array}{r} z+\sum_{k=2}^{\infty}\left[(k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n} a_k z^k \\ +(-1)^{n}\sum_{k=1}^{\infty}\left[(k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n}\overline{b_k z^k}\end{array}\right\} \\
+\left(1+e^{it}\right)\left\{\begin{array}{r} z+\sum_{k=2}^{\infty}\left[(k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n+1} a_k z^k \\ +(-1)^{n+1}\sum_{k=1}^{\infty}\left[(k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n+1}\overline{b_k z^k}\end{array}\right\}
\end{array}\right| \\
&- \left|\begin{array}{l}
\left(1+\beta+e^{it}\right)\left\{\begin{array}{r} z+\sum_{k=2}^{\infty}\left[(k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n} a_k z^k \\ +(-1)^{n}\sum_{k=1}^{\infty}\left[(k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n}\overline{b_k z^k}\end{array}\right\} \\
-\left(1+e^{it}\right)\left\{\begin{array}{r} z+\sum_{k=2}^{\infty}\left[(k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n+1} a_k z^k \\ +(-1)^{n+1}\sum_{k=1}^{\infty}\left[(k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n+1}\overline{b_k z^k}\end{array}\right\}
\end{array}\right|
\end{aligned}$$

$$= \left| \begin{array}{l} (2-\beta)\, z + \displaystyle\sum_{k=2}^{\infty} \left\{ \begin{array}{r} \left(1-\beta-e^{it}\right) + \left(1+e^{it}\right) \times \\ \left[(k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right] \end{array} \right\} \left[(k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n} a_k z^k \\ \quad +(-1)^{n} \displaystyle\sum_{k=1}^{\infty} \left\{ \begin{array}{r} \left(1-\beta-e^{it}\right) - \left(1+e^{it}\right) \times \\ \left[(k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right] \end{array} \right\} \left[(k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n} \overline{b_k z^k} \end{array} \right|$$

$$- \left| \begin{array}{l} \quad \beta z + \displaystyle\sum_{k=2}^{\infty} \left\{ \begin{array}{r} \left(1+\beta+e^{it}\right) - \left(1+e^{it}\right) \times \\ \left[(k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right] \end{array} \right\} \left[(k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n} a_k z^k \\ +(-1)^{n} \displaystyle\sum_{k=1}^{\infty} \left\{ \begin{array}{r} \left(1+\beta+e^{it}\right) + \left(1+e^{it}\right) \times \\ \left[(k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right] \end{array} \right\} \left[(k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n} \overline{b_k z^k} \end{array} \right|$$

$$= \left| \begin{array}{l} (2-\beta)\, z + \displaystyle\sum_{k=2}^{\infty} \left\{ \begin{array}{r} \left(1-\beta-e^{it}\right) + \left(1+e^{it}\right) \times \\ \left[(k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right] \end{array} \right\} \left[(k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n} a_k z^k \\ \quad -(-1)^{n} \displaystyle\sum_{k=1}^{\infty} \left\{ \begin{array}{r} \left(\beta-1+e^{it}\right) + \left(1+e^{it}\right) \times \\ \left[(k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right] \end{array} \right\} \left[(k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n} \overline{b_k z^k} \end{array} \right|$$

$$- \left| \begin{array}{l} \beta z - \displaystyle\sum_{k=2}^{\infty} \left\{ \begin{array}{r} \left(-1-\beta-e^{it}\right) + \left(1+e^{it}\right) \times \\ \left[(k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right] \end{array} \right\} \left[(k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n} a_k z^k \\ +(-1)^{n} \displaystyle\sum_{k=1}^{\infty} \left\{ \begin{array}{r} \left(1+\beta+e^{it}\right) + \left(1+e^{it}\right) \times \\ \left[(k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right] \end{array} \right\} \left[(k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n} \overline{b_k z^k} \end{array} \right|$$

$$\geq 2(1-\beta)|z| - \sum_{k=2}^{\infty} \left\{ \begin{array}{r} \left[4k-2\beta-2+4(k-1)-2+(\mu w^{\lambda}-\alpha)\right] \times \\ \left[(k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n} \end{array} \right\} |a_k||z|^k$$

$$- \sum_{k=1}^{\infty} \left\{ \begin{array}{r} \left[4k+2\beta+2+4(k+1)-2+(\mu w^{\lambda}-\alpha)\right] \times \\ \left[(k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n} \end{array} \right\} |b_k||z|^k$$

$$> 2(1-\beta) \left\{ \begin{array}{l} 1 - \displaystyle\sum_{k=2}^{\infty} \frac{\left[2\left((k-1)(\mu w^{\lambda}-\alpha)+k\right)-(1+\beta)\right]\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n}}{1-\beta} |a_k| \\ \quad - \displaystyle\sum_{k=1}^{\infty} \frac{\left[2\left((k+1)(\mu w^{\lambda}-\alpha)+k\right)+(1+\beta)\right]\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n}}{1-\beta} |b_k| \end{array} \right\}$$

This last expression is non-negative by (6), and so the proof is completed.

Theorem II.2 *Let* $f_n = h + \overline{g_n}$ *be given by (5). Then* $f_n \in \overline{S_H}\left(\lambda, w, n, \alpha, \beta\right)$ *if and only if*

$$\begin{aligned} &\sum_{k=2}^{\infty} \left\{ \left[2\left((k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right)-(1+\beta)\right]\left[(k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n} \right\} a_k \\ &+ \sum_{k=1}^{\infty} \left\{ \left[2\left((k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right)+(1+\beta)\right]\left[(k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n} \right\} b_k \\ &\leq (1-\beta)\,, \end{aligned} \tag{8}$$

where $\mu, \lambda, w \geq 0, 0 \leq \alpha \leq \mu w^{\lambda}$, $t \in R$, $n \in N_0$, $0 \leq \beta < 1$.

Proof: The "if" part follows from theorem (6) upon nothing that $\overline{S_H}\left(\lambda, w, n, \alpha, \beta\right) \subset S_H\left(\lambda, w, n, \alpha, \beta\right)$. For the "only if" part, we show that $f \notin \overline{S_H}\left(\lambda, w, n, \alpha, \beta\right)$ if the condition (8) does not hold. We note that a necessary and sufficient condition for $f_n = h + \overline{g_n}$ given by (5), to be in $\overline{S_H}\left(\lambda, w, n, \alpha, \beta\right)$ is that the condition (4) to be satisfied. This is equivalent to,

$$Re\left\{\frac{(1-\beta)z-\left[\begin{array}{l}\displaystyle\sum_{k=2}^{\infty}\begin{pmatrix}\left[2\left((k-1)(\mu w^{\lambda}-\alpha)+k\right)-(1+\beta)\right]\\ \times\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n}\end{pmatrix}a_k z^k\\ \displaystyle+\sum_{k=1}^{\infty}\begin{pmatrix}\left[2\left((k+1)(\mu w^{\lambda}-\alpha)+k\right)+(1+\beta)\right]\\ \times\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n}\end{pmatrix}b_k \bar{z}^k\end{array}\right]}{z-\displaystyle\sum_{k=2}^{\infty}\left[(k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n}a_k z^k+(-1)^n\sum_{k=1}^{\infty}\left[(k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n}b_k\bar{z}^k}\right\}\geq 0.$$

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing thae value of z on the opposite real axis where $0 \leq z = r < 1$ we must have

$$Re\left\{\frac{(1-\beta)-\left[\begin{array}{l}\displaystyle\sum_{k=2}^{\infty}\begin{pmatrix}\left[2\left((k-1)(\mu w^{\lambda}-\alpha)+k\right)-(1+\beta)\right]\\ \times\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n}\end{pmatrix}a_k r^{k-1}\\ \displaystyle+\sum_{k=1}^{\infty}\begin{pmatrix}\left[2\left((k+1)(\mu w^{\lambda}-\alpha)+k\right)+(1+\beta)\right]\\ \times\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n}\end{pmatrix}b_k r^{k-1}\end{array}\right]}{1-\displaystyle\sum_{k=2}^{\infty}\left[(k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n}a_k r^k+\sum_{k=1}^{\infty}\left[(k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n}b_k r^{k-1}}\right\}\geq 0. \tag{9}$$

If the condition (8) does not hold, then the numerator in (9) is negative for r sufficiently close to 1. Hence there exit $z_0 = r_0$ in (0,1) for which quotient in (9) is negative. This contradicts the required condition for $\overline{S_H}\left(\lambda, w, n, \alpha, \beta\right)$ and so the proof is complete.

III Distortion Inequalities and Extreme Points

In this section, we obtain extreme points for the class $\overline{S_H}\left(\lambda, w, n, \alpha, \beta\right)$.

Theorem III.1 *Let* $f_n \in \overline{S_H}\left(\lambda, w, n, \alpha, \beta\right)$. *Then for* $|z| = r < 1$ *we have*

$$|f_n(z)| \leq (1+b_1)r + \left\{\begin{array}{c}\dfrac{1-\beta}{\left[2(2+(\mu w^{\lambda}-\alpha))-(1+\beta)\right]\left[\mu w^{\lambda}-\alpha+2\right]^{n}}-\\ \dfrac{\left[2(2(\mu w^{\lambda}-\alpha+1))+(1+\beta)\right]\left[2(\mu w^{\lambda}-\alpha)+1\right]^{n}}{\left[2(2+(\mu w^{\lambda}-\alpha))-(1+\beta)\right]\left[\mu w^{\lambda}-\alpha+2\right]^{n}}b_1\end{array}\right\}r^2,$$

and

$$|f_n(z)| \geq (1-b_1)r - \left\{ \begin{array}{c} \dfrac{1-\beta}{[2(2+(\mu w^{\lambda}-\alpha))-(1+\beta)]\,[\mu w^{\lambda}-\alpha+2]^{n}} - \\ \dfrac{\left[2(2(\mu w^{\lambda}-\alpha+1))+(1+\beta)\right]\left[2(\mu w^{\lambda}-\alpha)+1\right]^{n}}{[2(2+(\mu w^{\lambda}-\alpha))-(1+\beta)]\,[\mu w^{\lambda}-\alpha+2]^{n}} b_1 \end{array} \right\} r^2.$$

Proof: We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f_n \in \overline{S_H}(\lambda, w, n, \alpha, \beta)$. Taking the absolute value of f_n, we have

$$\begin{aligned}
|f_n(z)| &\leq (1+b_1)r + \sum_{k=2}^{\infty}(a_k+b_k)r^k \\
&\leq (1+b_1)r + \sum_{k=2}^{\infty}(a_k+b_k)r^2 \\
&= (1+b_1)r + \left\{ \begin{array}{l} \dfrac{(1-\beta)r^2}{[2(2+(\mu w^{\lambda}-\alpha))-(1+\beta)]\,[\mu w^{\lambda}-\alpha+2]^{n}} \times \\ \displaystyle\sum_{k=2}^{\infty} \dfrac{\left[2(2+(\mu w^{\lambda}-\alpha))-(1+\beta)\right]\left[\mu w^{\lambda}-\alpha+2\right]^{n}}{1-\beta}\,[a_k+b_k] \end{array} \right\} \\
&\leq (1+b_1)r + \frac{(1-\beta)r^2}{[2(2+(\mu w^{\lambda}-\alpha))-(1+\beta)]\,[\mu w^{\lambda}-\alpha+2]^{n}} \times
\end{aligned}$$

$$\sum_{k=2}^{\infty} \left\{ \begin{array}{l} \dfrac{\left[2\left((k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right)-(1+\beta)\right]\left[(k-1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n}}{1-\beta} a_k + \\ \dfrac{\left[2\left((k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right)+(1+\beta)\right]\left[(k+1)\left(\mu w^{\lambda}-\alpha\right)+k\right]^{n}}{1-\beta} b_k \end{array} \right\}$$

$$\begin{aligned}
&\leq (1+b_1)r + \frac{(1-\beta)}{[2(2(\mu w^{\lambda}-\alpha))-(1+\beta)]\,[\mu w^{\lambda}-\alpha+2]^{n}} \times \\
&\left\{ 1 - \frac{\left[2(2(\mu w^{\lambda}-\alpha)+1)+(1+\beta)\right]\left[2(\mu w^{\lambda}-\alpha)+1\right]^{n}}{1-\beta} b_1 \right\} r^2 \\
&\leq (1+b_1)r + \left\{ \begin{array}{c} \dfrac{(1-\beta)}{[2(2(\mu w^{\lambda}-\alpha))-(1+\beta)]\,[\mu w^{\lambda}-\alpha+2]^{n}} - \\ \dfrac{\left[2(2(\mu w^{\lambda}-\alpha)+1)+(1+\beta)\right]\left[2(\mu w^{\lambda}-\alpha)+1\right]^{n}}{[2(2(\mu w^{\lambda}-\alpha))-(1+\beta)]\,[\mu w^{\lambda}-\alpha+2]^{n}} b_1 \end{array} \right\} r^2.
\end{aligned}$$

The following covering result follows from the left hand inequality in Theorem 3.1.

Corollary III.1 *Let f_n of the form(5) be so that $f_n \in \overline{S_H}(\lambda, w, n, \alpha, \beta)$. Then*

$$\left\{ w : |w| < \left[\frac{\begin{array}{c} \left[2(\mu w^{\lambda}-\alpha+2)-(1+\beta)\right]\left[\mu w^{\lambda}-\alpha+2\right]^{n}-(1+\beta)- \\ \left(\begin{array}{c} \left[2(\mu w^{\lambda}-\alpha+2)-(1+\beta)\right]\left[\mu w^{\lambda}-\alpha+2\right]^{n}-(1+\beta) \\ \times\left[2(2(\mu w^{\lambda}-\alpha)+1)+(1+\beta)\right]\left[2(\mu w^{\lambda}-\alpha)+1\right]^{n} \end{array} \right) \end{array}}{[2(\mu w^{\lambda}-\alpha+2)-(1+\beta)]\,[\mu w^{\lambda}-\alpha+2]^{n}} \right] \right\} \subset f_n(U).$$

Theorem III.2 *Let f_n begiven by (5). Then $f_n \in \overline{S_H}(\lambda, w, n, \alpha, \beta)$ if and only if*

$$f_n(z) = \sum_{k=1}^{\infty} \left[X_k h_k(z) + Y_k g_{nk}(z)\right], \qquad where, \quad h_1(z) = z, \tag{10}$$

$$h_k(z) = z - \left\{ \frac{1-\beta}{\left[2((k-1)(\mu w^{\lambda} - \alpha) + k) - (1+\beta)\right]\left[(k-1)(\mu w^{\lambda} - \alpha) + k\right]^n} z^k \right\}, k \geq 2,$$

$$g_{nk}(z) = z + (-1)^n \left\{ \frac{1-\beta}{\begin{array}{c}\left[2((k+1)(\mu w^{\lambda} - \alpha) + k) + (1+\beta)\right] \\ \times \left[(k+1)(\mu w^{\lambda} - \alpha) + k\right]^n\end{array}} z^{-k} \right\}, k \geq 2,$$

$$\sum_{k=1}^{\infty} \left[X_k + Y_k\right] = 1, \qquad X_k \geq 0, \qquad Y_k \geq 0.$$

In perticular, the extreme points of $\overline{S_H}(\lambda, w, n, \alpha, \beta)$ are $\{h_k\}$ and $\{g_n\}$.

Proof: For function f_n of the form (10) we may write,

$$\begin{aligned} f_n(z) &= \sum_{k=1}^{\infty} \left[X_k h_k(z) + Y_k g_{nk}(z)\right] \\ &= \sum_{k=1}^{\infty} \left[X_k + Y_k\right] z - \sum_{k=2}^{\infty} \left\{ \frac{1-\beta}{\left[2((k-1)(\mu w^{\lambda} - \alpha) + k) - (1+\beta)\right]\left[(k-1)(\mu w^{\lambda} - \alpha) + k\right]^n} X_k z^k \right\} \\ &\quad + (-1)^n \sum_{k=1}^{\infty} \left\{ \frac{1-\beta}{\left[2((k+1)(\mu w^{\lambda} - \alpha) + k) + (1+\beta)\right]\left[(k+1)(\mu w^{\lambda} - \alpha) + k\right]^n} Y_k z^{-k} \right\}. \\ Then&, \end{aligned}$$

$$\begin{aligned} &\sum_{k=2}^{\infty} \left\{ \begin{array}{c} \left(\frac{\left[2((k-1)(\mu w^{\lambda} - \alpha) + k) - (1+\beta)\right]\left[(k-1)(\mu w^{\lambda} - \alpha) + k\right]^n}{1-\beta} \right) \times \\ \left(\frac{1-\beta}{\left[2((k-1)(\mu w^{\lambda} - \alpha) + k) - (1+\beta)\right]\left[(k-1)(\mu w^{\lambda} - \alpha) + k\right]^n} X_k \right) \end{array} \right\} + \\ &\sum_{k=1}^{\infty} \left\{ \begin{array}{c} \left(\frac{\left[2((k+1)(\mu w^{\lambda} - \alpha) + k) + (1+\beta)\right]\left[(k+1)(\mu w^{\lambda} - \alpha) + k\right]^n}{1-\beta} \right) \times \\ \left(\frac{1-\beta}{\left[2((k+1)(\mu w^{\lambda} - \alpha) + k) + (1+\beta)\right]\left[(k+1)(\mu w^{\lambda} - \alpha) + k\right]^n} Y_k \right) \end{array} \right\} \\ &\sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leq 1, \end{aligned}$$

so $f_n \in \overline{S_H}(\lambda, w, n, \alpha, \beta)$.
Conversely, if $f_n \in \overline{S_H}(\lambda, w, n, \alpha, \beta)$, then

$$a_k \leq \frac{1-\beta}{[2((k-1)(\mu w^{\lambda}-\alpha)+k)-(1+\beta)]\,[(k-1)(\mu w^{\lambda}-\alpha)+k]^n}$$

$$and$$

$$b_k \leq \frac{1-\beta}{[2((k+1)(\mu w^{\lambda}-\alpha)+k)+(1+\beta)]\,[(k+1)(\mu w^{\lambda}-\alpha)+k]^n}$$

$$setting$$

$$X_k = \frac{\left[2((k-1)(\mu w^{\lambda}-\alpha)+k)-(1+\beta)\right]\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta}a_k; (k \geq 2),$$

$$Y_k = \frac{\left[2((k+1)(\mu w^{\lambda}-\alpha)+k)+(1+\beta)\right]\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta}b_k; (k \geq 1),$$

$$and$$

$$X_1 = 1 - \left(\sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k\right)$$

$$where, \quad X_1 \geq 0.$$

$$Then, \quad f_n(z) = X_1 z + \sum_{k=2}^{\infty} X_k h_k(z) + \sum_{k=1}^{\infty} Y_k g_{nk}(z)$$

so the proof is complete.

IV Convex Combination

In this section, we illustrate that the class $\overline{S_H}(\lambda, w, n, \alpha, \beta)$ is closed with regard to convex combination of its members.

Theorem IV.1 *The class $\overline{S_H}(\lambda, w, n, \alpha, \beta)$ is closed under convex combinations.*

Proof: Let $f_{n_i} \in \overline{S_H}(\lambda, w, n, \alpha, \beta)$ for $i = 1, 2, ...$, where f_{n_i} is given by

$$f_{n_i}(z) = z - \sum_{k=2}^{\infty} a_{k_i} z^k + (-1)^n \sum_{k=1}^{\infty} b_{k_i} \overline{z}^k.$$

Then by (7),

$$\sum_{k=2}^{\infty}\left\{\frac{\left[2((k-1)(\mu w^{\lambda}-\alpha)+k)-(1+\beta)\right]\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta}\right\}a_{k_i}+$$
$$\sum_{k=1}^{\infty}\left\{\frac{\left[2((k+1)(\mu w^{\lambda}-\alpha)+k)+(1+\beta)\right]\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta}\right\}b_{k_i} \leq 1. \tag{11}$$

For $\sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combinition of f_{n_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{n_i}(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{k_i} \right) z^k + (-1)^n \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i b_{k_i} \right) z^{-k}.$$

Then by (11),

$$\sum_{k=2}^{\infty} \frac{\left[2((k-1)(\mu w^{\lambda} - \alpha) + k) - (1+\beta)\right] \left[(k-1)(\mu w^{\lambda} - \alpha) + k\right]^n}{1-\beta} \left(\sum_{i=1}^{\infty} t_i a_{k_i} \right)$$
$$\sum_{k=1}^{\infty} \frac{\left[2((k+1)(\mu w^{\lambda} - \alpha) + k) + (1+\beta)\right] \left[(k+1)(\mu w^{\lambda} - \alpha) + k\right]^n}{1-\beta} \left(\sum_{i=1}^{\infty} t_i b_{k_i} \right)$$
$$= \sum_{i=1}^{\infty} t_i \left(\begin{array}{l} \displaystyle\sum_{k=2}^{\infty} \frac{\left[2((k-1)(\mu w^{\lambda} - \alpha) + k) - (1+\beta)\right] \left[(k-1)(\mu w^{\lambda} - \alpha) + k\right]^n}{1-\beta} a_{k_i} + \\ \displaystyle\sum_{k=1}^{\infty} \frac{\left[2((k+1)(\mu w^{\lambda} - \alpha) + k) + (1+\beta)\right] \left[(k+1)(\mu w^{\lambda} - \alpha) + k\right]^n}{1-\beta} b_{k_i} \end{array} \right)$$
$$\leq \sum_{i=1}^{\infty} t_i = 1.$$

This is the condition required by (7) and so $\sum_{i=1}^{\infty} t_i f_{n_i}(z) \in \overline{S_H}\,(\lambda, w, n, \alpha, \beta)$.

V CONCLUSION

In this paper an attempt has been made to introduce and investigate some properties for a new subclass of harmonic univalent functions by using a new differential operator. Based on this work, further useful study on different subclasses of harmonic univalent functions can be established.

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