# Irrational Behaviour of Bilateral Mock Theta Functions of Order Seven at Infinite Number of Points

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**Abstract** At the beginning of 1920, three months before his demise, Ramanujan addressed mock theta functions in his last letter to G. H. Hardy. He presented seventeen examples, and assigned them to orders three, five, and seven. Shukla and Ahmad obtained eight bilateral mock theta functions of order seven in 2003, and they proved that these functions satisfy the characteristic property of the mock theta function. In this paper it has been shown that four bilateral mock theta functions of order seven are irrational at $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$, and four are irrational at $q = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$.
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## 1 Introduction

Ramanujan introduced mock theta functions first time in his last letter to G.H. Hardy in January 1920 three months before his demise. In this letter, he remarked, "I discovered very interesting functions recently which I call 'Mock Theta Functions'...". He did not provide a definition of mock theta functions; instead, he provided a list of 17 examples and a qualitative explanation of the significant feature he had noticed. These 17 examples were designated as mock theta functions of orders three, five, and seven by Ramanujan. The mock theta functions of orders three and five were thoroughly studied by Watson [13, 14], who also added three more functions of order three to this collection.

Based on Ramanujan's qualitative explanation and subsequent observations made by other mathematicians as time to time, it may be said that:

A function defined by a $q$-series that is convergent for $|q| < 1$ and meets the following two requirements is called a *mock theta function*:

(i) For every root of unity $\xi$, there is a theta function $\theta_\xi(q)$ such that the difference $f(q) - \theta_\xi(q)$ is bounded as $q \to \xi$ radially,

(ii) There is no single $\theta$ function which works for all $\xi$, i.e. there is some root of unity $\xi$ for which $f(q) - \theta_\xi(q)$ is unbounded as $q \to \xi$ radially.

The *complete or bilateral mock theta function* that corresponds to a mock theta function like

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n}$$

is defined as

$$f_{0C}(q) = \sum_{n=-\infty}^{\infty} \frac{q^{n^2}}{(-q)_n}.$$

Ramanujan's mock theta functions of order three, Watson's three additional mock theta functions of order three, the Rogers-Ramanujan $q$-series, and six mock theta functions of order five exhibit irrational values at the points $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$, as showed by Mingarelli [5]. In 1999, Srivastava[12] obtained eight bilateral mock theta functions of order 'five' and provided some alternate forms for these functions. Ahmad[1] obtained eight bilateral mock theta functions of order 'nine' in 2015. Singh [9, 10] recently demonstrated that these bilateral mock theta functions of order five and nine are irrational at $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$.

Using the Slater [11] transformation (1.1.3) for $r = 3$, Shukla and Ahmad [7, 8] obtained eight bilateral mock theta functions of order 'seven' in 2003. They proved that these functions meet the distinctive quality of the mock theta function and they are the limiting cases of the basic hypergeometric series ${}_4\Phi_3$. Shukla and Ahmad's bilateral mock theta functions of order 'seven' are

$$f_0c_3(q) = \sum_{-\infty}^{\infty} \frac{(-1)^n q^{3\frac{(n^2-n)}{2}} q^n}{(-q;q)_n}, \tag{1}$$

$$f_1c_3(q) = \sum_{-\infty}^{\infty} \frac{(-1)^n q^{3\frac{(n^2-n)}{2}} q^{2n}}{(-q;q)_n}, \tag{2}$$

$$F_0c_3(q) = \sum_{-\infty}^{\infty} \frac{(-1)^n q^{3(n^2-n)} q^{2n}}{(q;q^2)_n}, \tag{3}$$

$$F_1c_3(q^2) = \sum_{-\infty}^{\infty} \frac{(-1)^n q^{3(2n^2-2n)} q^{8n}}{(q^6;q^4)_n}, \tag{4}$$

$$\Phi_0c_3(q) = \sum_{-\infty}^{\infty} \frac{(-1)^n q^{3n^2}}{(-q;q^2)_n}, \tag{5}$$

$$\Phi_1c_3(q) = \sum_{-\infty}^{\infty} (-1)^n q^{2n^2+4n} \left(-q;q^2\right)_n, \tag{6}$$

$$\Psi_0 c_3(q) = \sum_{-\infty}^{\infty} (-1)^n q^{n^2+3n} (-q;q)_n \tag{7}$$

and

$$\Psi_1 c_3(q) = \sum_{-\infty}^{\infty} \frac{(-1)^{n+1} q^{\frac{3n(n+1)}{2}}}{2(-q;q)_n}. \tag{8}$$

The irrational behaviour of the aforementioned functions has been demonstrated at infinite number of points in the present paper. To demonstrate the irrational behaviour of these functions, we require the Cantor series and theorems of Oppenheim [6] on the irrationality of a number given by the Cantor series, which are given in the Preliminaries. In subsections 3.1 and 3.2, the theorems for the irrational behaviour of functions (1) to (8) have been presented and shown.

# 2 Preliminaries

The Cantor series and the Oppenheim theorems to determine the irrationality of a given number are given in this section. Cantor gave [2] a criteria for determining the irrationality of a number $S$ given by the infinite series (Cantor series)

$$S = \sum_{n=1}^{\infty} \frac{b_n}{a_1 a_2 \cdots a_n}\ , \tag{9}$$

where the $a_i,\ b_i$ are integers satisfying the conditions $a_i \geq 2,\ a_i - 1 \geq b_i \geq 0$ and for every integer $k \geq 1$ there is an $n$ such that $k \mid a_1 a_2 \cdots a_n$. Cantor showed that $S$ is irrational if and only if the $b_i > 0$ infinitely often and $a_i - 1 > b_i$ infinitely often.

Oppenheim [6] and Diananda-Oppenheim [3] dropped the requirement that the product of the first n $a$'s be divisible and extended the theorem to the case in which the $b_i$ might have both signs. A publication by Hančl and Tijdeman [4] provided further development of the theory by avoiding the application of the Cantor-Oppenheim a priori requirement $a_i - 1 \geq b_i$. We utilised the notations as in [4] while $a$'s and $b$'s are interchanged in [6], [3]. Oppenheim's two following theorems about the Cantor series have been utilised for showing how bilateral mock theta functions of order seven behave at an infinite number of points:

**Theorem 1 ([6], Theorem 4)** *Let $(a_n),\ (b_n)$ be two sequences of integers with $a_n \geq 2,\ 0 \leq b_n \leq a_n - 1$. If $b_n > 0$ infinitely often and if there is a subsequence $i_n$ such that $a_{i_n} \to \infty$ and $b_{i_n}/a_{i_n} \to 0$ as $n \to \infty$, then $S$ as defined in (9) is irrational.*

**Theorem 2 ([6], Theorem 8)** *Let $(a_n),\ (b_n)$ be two sequences of integers with $a_n \geq 2,\ |b_n| \leq a_n - 1$. Furthermore, let $b_m b_n < 0$ for some $m > i,\ n > i$ for any assigned integer $i$. If there is a subsequence $i_n$ such that $a_{i_n} \to \infty$ and $b_{i_n}/a_{i_n} \to 0$ as $n \to \infty$, then $S$ given by (9) is irrational.*

Along with some standard results, the following $q$-notations are also used in this paper:

For $|q| < 1$,

$$(a)_n = (a;q)_n = (1-a)(1-aq)\cdots\left(1-aq^{n-1}\right), \qquad (a;q)_0 = 1,$$

$$(a_1, a_2, \cdots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n .$$

The basic hypergeometric series with base $q$ is

$${}_r\Phi_{r-1}\left[\begin{array}{c} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_{r-1} \end{array}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_r; q)_n}{(q, b_1, b_2, \cdots, b_{r-1}; q)_n} z^n$$

for $|z| < 1$.

The bilateral basic hypergeometric series with base $q$ is

$${}_r\Psi_r\left[\begin{array}{c} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_r \end{array}; q, z\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \cdots, a_r; q)_n}{(b_1, b_2, \cdots, b_r; q)_n} z^n$$

for $\left|\frac{b_1 b_2 \cdots b_r}{a_1 a_2 \cdots a_r}\right| < |z| < 1$,

$$\text{and} \qquad (a; q^k)_{-n} = \frac{(-q^k/a)^n}{(q^k/a; q^k)_n} q^{\frac{kn(n-1)}{2}}.$$

# 3 Main results

In this section, we shall prove two theorems for the irrational behaviour of bilateral mock theta functions of order seven, first one is Theorem 3 for the functions $F_0c_3(q)$, $F_1c_3(q^2)$, $\Phi_0c_3(q)$ and $\Phi_1c_3(q)$ and second is Theorem 4 for $f_0c_3(q)$, $f_1c_3(q)$, $\Psi_0c_3(q)$ and $\Psi_1c_3(q)$.

## 3.1 Irrational Behaviour of the Functions (3), (4), (5) and (6)

**Theorem 3** *Bilateral mock theta functions* $F_0c_3(q)$, $F_1c_3(q^2)$, $\Phi_0c_3(q)$ *and* $\Phi_1c_3(q)$ *take on irrational values at* $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$.

*Proof* In order to prove the Theorem 3, we shall prove it separately for each function $F_0c_3(q)$, $F_1c_3(q^2)$, $\Phi_0c_3(q)$ and $\Phi_1c_3(q)$:

### 3.1.1 $F_0c_3(q)$ is irrational at $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$

The function (3) can alternatively be expressed as follows:

$$F_0c_3(q) = 1 + S_{F_{0,3}}(q) + \sum_{n=1}^{\infty} q^{2n^2+n} \left(q; q^2\right)_n . \tag{10}$$

From above expression, it is obvious that $F_0c_3(q)$ is irrational if and only if $S_{F_{0,3}}(q)$ is irrational, and

$$S_{F_{0,3}}(q) = \sum_{n=1}^{\infty} \frac{(-1)^n q^{3n^2-n}}{(1-q)(1-q^3)\cdots(1-q^{2n-1})}. \tag{11}$$

If $p, q$ be two complex numbers and $q \neq 0$, then

$$S_{F_{0,3}}\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n p^{3n^2-n} q^n}{q^{2n^2}(q-p)(q^3-p^3)\cdots(q^{2n-1}-p^{2n-1})}. \tag{12}$$

Considering $p = 1$ in the above sum, we obtain

$$S_{F_{0,3}}\left(\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{q^2(q-1)\,q^6(q^3-1)\cdots q^{4n-2}(q^{2n-1}-1)}. \tag{13}$$

The above sum is a Cantor series, as provided by (9) with the identifications $b_n = (-1)^n q^n$ and $a_n = q^{4n-2}(q^{2n-1}-1)$ for all $n \geq 1$. For every integer $q \geq 2$ and $n \geq 1$, $a_n \geq 2,\ a_n - 1 > |b_n| > 0,\ a_n \to \infty,\ b_n/a_n \to 0$ as $n \to \infty$. Consequently, according to Theorem 2, $S_{F_{0,3}}\left(\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$.

Applying $p = -1$ in (12) once again, we have

$$S_{F_{0,3}}\left(-\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{q^2(q+1)\,q^6(q^3+1)\cdots q^{4n-2}(q^{2n-1}+1)}. \tag{14}$$

In accordance with (9) $S_{F_{0,3}}\left(-\frac{1}{q}\right)$ is a Cantor series with

$$b_n = (-1)^n q^n \quad \text{and} \quad a_n = q^{4n-2}\left(q^{2n-1}+1\right)$$

for all $n$. Since every requirement of Theorem 2 has been fulfilled, it follows that $S_{F_{0,3}}\left(-\frac{1}{q}\right)$ is irrational for any integer $q \geq 2$.

Thus, $S_{F_{0,3}}(q)$ is irrational for $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$ as $S_{F_{0,3}}\left(\frac{1}{q}\right)$ and $S_{F_{0,3}}\left(-\frac{1}{q}\right)$ are irrational for all integers $q \geq 2$.

#### 3.1.2 $F_1c_3(q^2)$ is irrational at $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$

Another way to state the function (4) is as follows:

$$F_1c_3\left(q^2\right) = 1 + S_{F_{1,3}}(q) + \sum_{n=1}^{\infty} q^{4n^2+2n}\left(q^{-2};q^4\right)_n. \tag{15}$$

where

$$S_{F_{1,3}}(q) = \sum_{n=1}^{\infty} \frac{(-1)^n q^{6n^2+2n}}{(1-q^6)(1-q^{10})\cdots(1-q^{4n+2})} \tag{16}$$

It is apparent that if and only if $S_{F_{1,3}}(q)$ is irrational, then $F_1c_3(q^2)$ is also irrational. As previously, for complex numbers $p, q,\ q \neq 0$, we have

$$S_{F_{1,3}}\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n p^{6n^2+2n} q^{2n}}{q^{4n^2}(q^6-p^6)(q^{10}-p^{10})\cdots(q^{4n+2}-p^{4n+2})}. \tag{17}$$

When $p = \pm 1$ is substituted in above sum, we yield

$$S_{F_{1,3}}\left(\pm\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n q^{2n}}{q^4(q^6-1)\,q^{12}(q^{10}-1)\cdots q^{8n-4}(q^{4n+2}-1)}. \tag{18}$$

When compared to (9), the aforementioned sum is a Cantor series, as shown by the identifications $b_n = (-1)^n q^{2n}$ and $a_n = q^{8n-4}(q^{4n+2}-1)$. All of the requirements of Theorem 2 are fulfilled for every integer $q \geq 2$ and every $n \geq 1$, meaning that $S_{F_{1,3}}\left(\pm\frac{1}{q}\right)$ is irrational for any integer $q \geq 2$. Therefore, for $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$, $S_{F_{1,3}}(q)$ is irrational..

### 3.1.3 $\Phi_0 c_3(q)$ is irrational at $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$

Following is also a method of expressing the function (5):

$$\Phi_0 c_3 (q) = 1 + S_{\Phi_{0,3}} (q) + \sum_{n=1}^{\infty} (-1)^n q^{2n^2} \left(-q; q^2\right)_n \tag{19}$$

where

$$S_{\Phi_{0,3}} (q) = \sum_{n=1}^{\infty} \frac{(-1)^n q^{3n^2}}{(1+q)(1+q^3)\cdots(1+q^{2n-1})}. \tag{20}$$

It follows that if and only if $S_{\Phi_{0,3}} (q)$ is irrational, $\Phi_0 c_3 (q)$ is also irrational.

For complex numbers $p, q,\ q \neq 0$, we see that

$$S_{\Phi_{0,3}} \left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n p^{3n^2}}{q^{2n^2} (q+p)(q^3+p^3)\cdots(q^{2n-1}+p^{2n-1})}. \tag{21}$$

Now, if we put $p = 1$ in (21), we find

$$S_{\Phi_{0,3}} \left(\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{q^2 (q+1) q^6 (q^3+1) \cdots q^{4n-2} (q^{2n-1}+1)}. \tag{22}$$

The sum $S_{\Phi_{0,3}} \left(\frac{1}{q}\right)$ is a Cantor series as per (9) with $b_n = (-1)^n$ and $a_n = q^{4n-2} (q^{2n-1}+1)$ for every $n \geq 1$. Furthermore, all of the requirements of Theorem 2 are fulfilled for every integer $q \geq 2$ and any $n \geq 1$, so $S_{\Phi_{0,3}} \left(\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$.

Setting $p = -1$ in (21) yields

$$S_{\Phi_{0,3}} \left(-\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{1}{q^2 (q-1) q^6 (q^3-1) \cdots q^{4n-2} (q^{2n-1}-1)}. \tag{23}$$

The sum $S_{\Phi_{0,3}} \left(-\frac{1}{q}\right)$ is a Cantor series as per (9) with $a_n = q^{4n-2} (q^{2n-1}-1)$ and $b_n = 1$ for every $n \geq 1$. Thus, according to the Theorem 1, $S_{\Phi_{0,3}} \left(-\frac{1}{q}\right)$ is irrational for any integer $q \geq 2$.

It has been observed that $S_{\Phi_{0,3}} \left(\frac{1}{q}\right)$ and $S_{\Phi_{0,3}} \left(-\frac{1}{q}\right)$ are irrational for every integer $q \geq 2$. Hence $S_{\Phi_{0,3}} (q)$ is irrational for $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$.

### 3.1.4 $\Phi_1 c_3(q)$ is irrational at $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$

The following is another approach to state the function (6):

$$\Phi_1 c_3 (q) = \sum_{n=0}^{\infty} (-1)^n q^{2n^2+4n} \left(-q; q^2\right)_n + S_{\Phi_{1,3}} (q). \tag{24}$$

It is evident that $\Phi_1 c_3 (q)$ is irrational if and only if $S_{\Phi_{1,3}} (q)$ is irrational, and

$$S_{\Phi_{1,3}}(q) = \sum_{n=1}^{\infty} \frac{(-1)^n q^{3n^2-4n}}{(1+q)(1+q^3)\cdots(1+q^{2n-1})}. \tag{25}$$

Similar to before, with complex numbers $p, q\ q \neq 0$

$$S_{\Phi_{1,3}}\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n p^{3n^2-4n} q^{4n}}{q^{2n^2}\left(q+p\right)\left(q^3+p^3\right)\cdots\left(q^{2n-1}+p^{2n-1}\right)}. \tag{26}$$

When $p = 1$, the above sum yields

$$\begin{aligned} S_{\Phi_{1,3}}\left(\frac{1}{q}\right) &= -\frac{q^2}{q+1} \\ &+ \frac{1}{q^2(q+1)} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{4(n+1)}}{q^6\left(q^3+1\right) q^{10}\left(q^5+1\right)\cdots q^{4n+2}\left(q^{2n+1}+1\right)}. \end{aligned} \tag{27}$$

If the sum on the right of (27) is irrational, it is evident that $S_{\Phi_{1,3}}\left(\frac{1}{q}\right)$ is irrational. Moreover, it is the Cantor series form as shown by (9) with the identifications $a_n = q^{4n+2}\left(q^{2n+1}+1\right)$ and $b_n = (-1)^{n+1} q^{4(n+1)}$ for all $n \geq 1$. Thus, the sum on the right of (27) is irrational for $q \geq 2$ because every integer with $q \geq 2$ and $n \geq 1$ satisfies all of Theorem 2's criteria. Consequently, $S_{\Phi_{1,3}}(\frac{1}{q})$ is irrational for $q \geq 2$.

When $p = -1$, the sum (26) yields

$$\begin{aligned} S_{\Phi_{1,3}}\left(-\frac{1}{q}\right) &= \frac{q^2}{q-1} \\ &+ \frac{1}{q^2(q-1)} \sum_{n=1}^{\infty} \frac{q^{4(n+1)}}{q^6\left(q^3-1\right) q^{10}\left(q^5-1\right)\cdots q^{4n+2}\left(q^{2n+1}-1\right)}. \end{aligned} \tag{28}$$

It is apparent that if the sum to the right of (28) is irrational, then $S_{\Phi_{1,3}}\left(-\frac{1}{q}\right)$ is also irrational. Furthermore, this sum is a Cantor series as seen by (9) with $a_n = q^{4n+2}\left(q^{2n-1}-1\right)$ and $b_n = q^{4(n+1)}$ for all $n \geq 1$. Since all of the requirements of Theorem 1 are fulfilled for every integer $q \geq 2$ and any $n \geq 1$, so the sum to the right of (28) is irrational. Thus, $S_{\Phi_{1,3}}\left(-\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$.

Due to the fact that $S_{\Phi_{1,3}}\left(\frac{1}{q}\right)$ and $S_{\Phi_{1,3}}\left(-\frac{1}{q}\right)$ are irrational for every integer $q \geq 2$, hence $S_{\Phi_{1,3}}\left(q\right)$ is irrational for $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$.

## 3.2 Irrational Behaviour of the Functions (1), (2), (7) and (8)

**Theorem 4** *Bilateral mock theta functions $f_0c_3(q)$, $f_1c_3(q)$, $\Psi_0c_3(q)$ and $\Psi_1c_3(q)$ take on irrational values at $q = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$.*

*Proof* We are now going to show the results of Theorem (4) for each of the functions $f_0c_3\left(q\right)$, $f_1c_3\left(q\right)$, $\Psi_0c_3\left(q\right)$ and $\Psi_1c_3\left(q\right))$ individually:

### 3.2.1 $f_0c_3(q)$ is irrational at $q = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$

Alternative form for the function (1) is

$$f_0c_3\left(q\right) = 1 + S_{f_{0,3}}\left(q\right) + \sum_{n=1}^{\infty} (-1)^n q^{n^2+n}\left(-1; q\right)_n. \tag{29}$$

If and only if $S_{f_{0,3}}(q)$ is irrational, it is apparent that $f_0c_3(q)$ is irrational, and

$$S_{f_{0,3}}(q) = \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{3n^2-n}{2}}}{(1+q)(1+q^2)\cdots(1+q^n)}. \tag{30}$$

For complex numbers $p, q,\ q \neq 0$, just like in the preceding section, we have

$$S_{f_{0,3}}\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n p^{\frac{3n^2-n}{2}} q^n}{q^{n^2}(q+p)(q^2+p^2)\cdots(q^n+p^n)}. \tag{31}$$

The above sum yields when $p = 1$,

$$S_{f_{0,3}}\left(\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{q(q+1)q^3(q^2+1)\cdots q^{2n-1}(q^n+1)}. \tag{32}$$

According to (9) the sum $S_{f_{0,3}}\left(\frac{1}{q}\right)$ is of the Cantor series form with $a_n = q^{2n-1}(q^n+1)$ and $b_n = (-1)^n q^n$. In the context of Theorem 2, $S_{f_{0,3}}\left(\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$, and hence $S_{f_{0,3}}(q)$ is irrational for $q = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$.

#### 3.2.2 $f_1c_3(q)$ is irrational at $q = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$

We can express the function (2) as:

$$f_1c_3(q) = 1 + S_{f_{1,3}}(q) + \sum_{n=1}^{\infty}(-1)^n q^{n^2}(-1;q)_n, \tag{33}$$

where

$$S_{f_{1,3}}(q) = \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{3n^2+n}{2}}}{(1+q)(1+q^2)\cdots(1+q^n)}. \tag{34}$$

Whenever it is defined, if $S_{f_{1,3}}(q)$ is irrational, then $f_1c_3(q)$ is also irrational. For complex numbers $p, q,\ q \neq 0$, we have

$$S_{f_{1,3}}\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n p^{\frac{3n^2+n}{2}}}{q^{n^2}(q+p)(q^2+p^2)\ldots(q^n+p^n)}. \tag{35}$$

The above sum gives when p = 1,

$$S_{f_{1,3}}\left(\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{q(q+1)q^3(q^2+1)\cdots q^{2n-1}(q^n+1)}. \tag{36}$$

The sum $S_{f_{1,3}}\left(\frac{1}{q}\right)$ has the Cantor series form (9) with $b_n = (-1)^n$ and $a_n = q^{2n-1}(q^n+1)$. Thus, $S_{f_{1,3}}\left(\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$ since Theorem 2's criteria are all satisfied, and consequently $S_{f_{1,3}}(q)$ is irrational for $q = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ also.

#### 3.2.3 $\Psi_0 c_3(q)$ is irrational at $q = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$

An alternate form of (7) is

$$\Psi_0 c_3 (q) = \sum_{n=0}^{\infty} (-1)^n q^{n^2+3n} (-q;q)_n + S_{\Psi_{0,3}} (q), \tag{37}$$

where

$$S_{\Psi_{0,3}} (q) = \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{3n^2-7n}{2}}}{(1+1)(1+q)(1+q^2)\cdots(1+q^{n-1})}. \tag{38}$$

If and only if $S_{\Psi_{0,3}}(q)$ is irrational, it is evident that $\Psi_0 c_3(q)$ is also irrational. Similar to the prior section, for complex numbers $p, q,\ q \neq 0$, we have

$$S_{\Psi_{0,3}} \left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n p^{\frac{3n^2-7n}{2}} q^{3n}}{q^{n^2}(1+1)(q+p)(q^2+p^2)\cdots(q^{n-1}+p^{n-1})}. \tag{39}$$

When $p = 1$ is taken in (39), we obtain

$$\begin{aligned} S_{\Psi_{0,3}} \left(\frac{1}{q}\right) &= -\frac{q^2}{2} + \frac{q^2}{2(q+1)} \\ &\quad + \frac{1}{2(q+1)} \sum_{n=1}^{\infty} \frac{(-1)^n q^2}{q^2(q^2+1)q^4(q^3+1)\cdots q^{2n}(q^{n+1}+1)}. \end{aligned} \tag{40}$$

If the sum to the right of (40) is irrational and this sum is a Cantor series as described in (9) with $a_n = q^{2n}(q^{n+1}+1)$ and $b_n = (-1)^n q^2$, then $S_{\Psi_{0,3}}\left(\frac{1}{q}\right)$ is evidently irrational. For every integer $q \geq 2$, $S_{\Psi_{0,3}}\left(\frac{1}{q}\right)$ is therefore irrational according to Theorem 2. Therefore, when $q = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$, $S_{\Psi_{0,3}}(q)$ is irrational.

#### 3.2.4 $\Psi_1 c_3(q)$ is irrational at $q = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$

The function (8) can be stated as follows:

$$\Psi_1 c_3 (q) = -\frac{1}{2} + \frac{1}{2} S_{\Psi_{1,3}} (q) + \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n+1} q^{n(n-1)} (-1;q)_n, \tag{41}$$

where

$$S_{\Psi_{1,3}} (q) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{\frac{3n(n+1)}{2}}}{(1+q)(1+q^2)\cdots(1+q^n)}. \tag{42}$$

It is obvious that $\Psi_1 c_3(q)$ is irrational if and only if $S_{\Psi_{1,3}}(q)$ is irrational. We have the following for complex numbers $p, q$ and $q \neq 0$,

$$S_{\Psi_{1,3}} \left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} p^{\frac{3n(n+1)}{2}}}{q^{n(n+1)}(q+p)(q^2+p^2)\cdots(q^n+p^n)}. \tag{43}$$

Given the preceding sum with $p = 1$, we find

$$S_{\Psi_{1,3}}\left(\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{q^2\,(q+1)\,q^4\,(q^2+1)\cdots q^{2n}\,(q^n+1)}. \tag{44}$$

As per (9), the sum $S_{\Psi_{1,3}}\left(\frac{1}{q}\right)$ is a Cantor series with $a_n = q^{2n}\,(q^n+1)$ and $b_n = (-1)^{n+1}$ for all $n \geq 1$. All of the criteria of Theorem 2 hold true for any integer $q \geq 2$ and any $n \geq 1$, thereby $S_{\Psi_{1,3}}\left(\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$. As a result,$S_{\Psi_{1,3}}(q)$ is irrational at $q = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$.

# 4 Conclusion

In Theorem 3, we demonstrated that bilateral mock theta functions of order seven $F_0c_3(q)$, $F_1c_3(q^2)$, $\Phi_0c_3(q)$ and $\Phi_1c_3\,(q)$ take on irrational values at $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$, while Theorem 4 demonstrated that the functions $f_0c_3(q)$, $f_1c_3(q)$, $\Psi_0c_3(q)$ and $\Psi_1c_3(q)$ take on irrational values at $q = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$. Because the Theorems 1 and 2 are not enough, we did not explore the irrational behaviour of the functions in Theorem 4 at $q = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \cdots$. If we have another criterion, then we might be able to speak up. By applying this idea, anyone can find the irrational behaviour of other infinite series.

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